Lecture 31

11.11 - Applications of Taylor Series

In Calc I, you did linear approximations of a function near a:

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for } x \text{ close to } a$$

This is approximating $f(x)$ by its first Taylor polynomial $T_1(x)$. This is sometimes called a <u>first order approximation</u> of f. As might be expected, if we take more terms of the Taylor series, we get a better approximation:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

near a. This is an <u>n^{th} order approximation</u>. Also, taking larger n usually allows us to expand the range of x values on which the approximation has an acceptable error.

See Mathematica code for approximation of e^x

When we approximate $f(x)$ by $T_n(x)$, the error is $|R_n(x)| = |f(x) - T_n(x)|$, which we have two possible ways to compute:

1) (Always Works) Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!} \quad \text{for } |x-a| \leq d.$$

2) If the Taylor series is an alternating series, we can use the error estimates from that:

$$S = \sum_{k=1}^{\infty} (-1)^{k-1} b_k \quad , \quad S_n = \sum_{k=1}^{n} (-1)^{k-1} b_k$$

$$|R_n| = |S - S_n| \leq b_{n+1}$$

Ex: Consider $f(x)=e^x$ and its Maclauren series.

ⓐ What is the 4^{th} order Taylor approximation of e^x?

ⓑ How accurate is this approximation on $[-4,4]$?

ⓒ Find an interval around 0 (e.g. $[-d,d]$) on which the error be made less than 10^{-3}.

Ex: Consider again $f(x) = e^x$.

ⓐ What is the 3^{rd} Taylor polynomial of $f(x)$ at $a = 2$?

ⓑ What is an upper bound on the error of approximating e^x with this on $[1, 3]$?

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Ex: ⓐ Approximate $f(x) = \cos x$ to 3^{rd} order at $a = \frac{\pi}{2}$.

ⓑ The Taylor series for $\cos x$ is alternating. Use the alternating series error estimation to find the maximum error on $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.